West Windsor-Plainsboro Regional School District
Multivariable Calculus Honors
July 2022

Unit 1: Vectors and the Geometry of Space	
Content Area: Mathematics	
Course & Grade Level: Multivariable Calculus Honors, grade 12	
Summary and Rationale	
In order to transition from calculus in the two-dimensional plane to calculus in three-dimensional space, mathematicians turn to vectors and parametric equations to describe curves and surfaces. Understanding vectors and their properties in both 2D and 3D is crucial for developing the foundation of multivariable calculus. The dot product and cross product are paramount in developing the equation for a line and the equation for a plane, which are the fundamental basis for curves and surfaces, respectively. Representations of geometric surfaces, such as cylinders, spheres, and cones are also established in 3D space.	
Recommended Pacing	
Approximately 16 days	
New Jersey Student Learning Standards for Mathematical Practice	
Standard: Standards for	
CPI #	**Cumulative Progress Indicator (CPI)**
1	Make sense of problems and persevere in solving them.
2	Reason abstractly and quantitatively.
3	Construct viable arguments and critique the reasoning of others.
4	Model with mathematics.
5	Use appropriate tools strategically.
6	Attend to precision.
7	Look for and make use of structure.
8	Look for and express regularity in repeated reasoning.
New Jersey Student Learning Standards for English Language Arts Companion Standards	
Standard: Science Key Ideas and Details	
CPI #	**Cumulative Progress Indicator (CPI)**
RST.9-10.3	Follow precisely a complex multistep procedure when carrying out experiments, taking measurements, or performing technical tasks, attending to special cases or exceptions defined in the text. Follow precisely a multistep procedure when carrying out experiments, taking measurements, or performing technical tasks.
New Jersey Student Learning Standards for Career Readiness, Life Literacies and Key Skills	
CPI #	**Cumulative Progress Indicator (CPI)**
9.4.12.CT.2	Explain the potential benefits of collaborating to enhance critical thinking and problem solving (e.g., 1.3E.12profCR3.a).
New Jersey Student Learning Standards for Computer Science and Design Thinking	
CPI #	**Cumulative Progress Indicator (CPI)**
8.2.12.NT.1	Explain how different groups can contribute to the overall design of a product.
Instructional Focus	
Unit Enduring Understandings	

- We live in a three-dimensional world, which requires mathematical models that involve multiple variables.
- Various multivariable functions can be explored and understood verbally, numerically, algebraically, and visually.
- Vector functions visually model behavior the same way in their 2D and 3D representations whereas scalar equations modeling behavior in 2D do not have the same visual representation in 3D.

Unit Essential Questions

- How can the concepts of multivariable calculus help us understand the physical world around us?
- How can a previous knowledge of single-variable calculus be used to understand the concepts of multivariable calculus?
- Why can the curves represented by real-value functions in 2D not be represented by real-value functions in 3D?

Content Understandings

- The visual representation of points and curves in the coordinate plane can be extended to visually represent points, lines, planes, and surfaces in space.
- Real-value functions of two variables are a representation of surfaces in space.
- Vectors have scalar components that allow vectors to be manipulated algebraically.
- Vectors can help define lines and planes in space.
- Vectors can describe position and motion of an object in space as well the forces acting upon the object.

Content Questions

- What is the distance between two points in space?
- What is a vector?
- How does one measure a vector's magnitude and direction?
- What is a dot product and how is it used in various applications?
- What is a cross product and how is it used in various applications?
- What are the differences between the algebraic and visual definitions of lines in space and the algebraic and visual definitions of surfaces?

Objectives

Students are learning to/that:

- Locate points in space and write their coordinates triples.
- Find distance between two points in space.
- Write an equation of a sphere, cone, and other 3D geometric shapes.
- Represent vectors geometrically and algebraically.
- Utilize the properties of vectors to simplify and solve problems.
- Determine the magnitude of a vector.
- Define a unit vector.
- Define a vector in the plane or in space using its direction and magnitude.
- Determine the dot product of two vectors.
- Find the interior angle between two vectors.
- Determine if two vectors in space are orthogonal.
- Find the projection of a vector.
- Define work as the dot product of a force vector and a distance vector.
- Find the cross product of two vectors.
- Prove algebraic properties of the cross product.
- Use geometric properties of the cross product.
- Determine a vector equation of a line in space.

- Determine parametric equations for a line in space.
- Write symmetric equations of a line in space.
- Determine whether two distinct lines are skew, parallel, or intersecting.
- Determine the point of intersection of two distinct lines in space.
- Find an equation of a plane.
- Determine whether two distinct planes are parallel or intersecting.
- Determine the line of intersection of two planes.
- Find the distance from a point to a plane and from a point to a line.
- Identify quadric surfaces based on an ellipse.
- Identify quadric surfaces based on a hyperbola.
- Identify cylinders.
- Graph quadric surfaces.

Evidence of Learning
Assessment
Assessment plan may include teacher designed formative and summative assessments, a district common assessment, analysis of standardized tests and NJSLA data.
Resources
Core Text: Multivariable Calculus, Early Transcendentals, 2e (2019), Sullivan & Miranda

Unit 2: Vector Functions	
Content Area: Mathematics	
Course & Grade Level: Multivariable Calculus Honors, grade 12	
Summary and Rationale	
In single variable calculus, 1D and 2D motion is explored using real-valued functions and parametric equations. To describe motion in 3D space, vector-valued functions are used. Vector-valued functions have components of real-valued functions, which allows calculus to be applied to these vector-valued functions in a similar fashion as real-valued functions. Limits, derivatives, and integrals of these vector-valued functions are explored in the context of motion in 3D space– position, displacement, distance, curvature, velocity, speed, and acceleration.	
Recommended Pacing	
Approximately 14 days	
New Jersey Student Learning Standards for Mathematical Practice	
Standard: Standards for	
CPI #	**Cumulative Progress Indicator (CPI)**
1	Make sense of problems and persevere in solving them.
2	Reason abstractly and quantitatively.
3	Construct viable arguments and critique the reasoning of others.
4	Model with mathematics.
5	Use appropriate tools strategically.
6	Attend to precision.
7	Look for and make use of structure.
8	Look for and express regularity in repeated reasoning.
New Jersey Student Learning Standards for English Language Arts Companion Standards	
Standard: Science Key Ideas and Details	
CPI #	**Cumulative Progress Indicator (CPI)**
RST.9-10.3	Follow precisely a complex multistep procedure when carrying out experiments, taking measurements, or performing technical tasks, attending to special cases or exceptions defined in the text. Follow precisely a multistep procedure when carrying out experiments, taking measurements, or performing technical tasks.
New Jersey Student Learning Standards for Career Readiness, Life Literacies and Key Skills	
CPI #	**Cumulative Progress Indicator (CPI)**
9.4.12.CT.2	Explain the potential benefits of collaborating to enhance critical thinking and problem solving (e.g., 1.3E.12profCR3.a).
New Jersey Student Learning Standards for Computer Science and Design Thinking	
CPI #	**Cumulative Progress Indicator (CPI)**
8.2.12.NT.1	Explain how different groups can contribute to the overall design of a product.
Instructional Focus	
Unit Enduring Understandings	
• Vector functions describe curves in space as a function of a parameter. • The limits, derivatives, and integrals of vectors describe motion of particles since their components are scalar functions.	

- Multivariable calculus concepts help explain phenomena in the physical world.

Unit Essential Questions

- How does one measure the length of a space curve?
- How do unit tangent, unit normal, and unit bi-normal vectors as well as normal and osculating planes relate to a space curve?
- How do the derivatives of a position vector function relate to the velocity and acceleration of a particle?

Content Understandings

- Vector-valued functions are functions whose domain is a subset of real numbers and whose range consists of vectors in the plane or space.
- Vector-valued functions can model various applications such as projectile motion, planetary motion, modeling DNA, and film animation.

Content Questions

- What is a vector function?
- What is a derivative and integral of a vector function?
- What is a space curve and how do we measure its length and curvature?
- How can the ideas of tangent and normal vectors and curvature be used in physics to study the motion of an object, including its velocity and acceleration, along a space curve?

Objectives

Students are learning to/that:

- Determine the domain of a vector function.
- Graph a vector function.
- Find the limit and determine the continuity of a vector function.
- Find the derivative of a vector function.
- Interpret the derivative of a vector function geometrically.
- Find the unit tangent vector and the principal unit normal vector of a smooth curve.
- Find the arc length of a curve traced out by a vector function.
- Determine whether the parameter used in a vector function is arc length.
- Find the curvature of a curve in the plane or in space.
- Find the curvature of a plane curve given by y = f(x).
- Find an osculating circle of a curve at a particular point.
- Find the velocity, acceleration, and speed of a moving particle.
- Express the acceleration vector using tangential and normal components.
- Integrate vector function.
- Solve projectile motion problems.

Evidence of Learning

Assessment

Assessment plan may include teacher designed formative and summative assessments, a district common assessment, analysis of standardized tests and NJSLA data.

Resources

Core Text: Multivariable Calculus, Early Transcendentals, 2e (2019), Sullivan & Miranda

Unit 3: Functions of Several Variables	
Content Area: Mathematics	
Course & Grade Level: Multivariable Calculus Honors, grade 12	
Summary and Rationale	
In single variable calculus, the use of limits to show continuity of a function leads to the rate of change of a function with respect to the change of its independent variable. Many functions often depend on two or more variables such as ambient temperature as a function of longitude and latitude. To describe the rate of change of a multivariable-function, a precise definition of a limit is explored and used to find partial derivatives. These derivatives are used to see that the rate of change of a multivariable function is a combination of the rates of change with respect to each independent variable of the function. Partial derivatives coupled with vectors lead to the understanding of directional derivatives and gradients. With these concepts established, we can optimize functions to find extrema in 3D similarly to the methods used in single-variable calculus.	
Recommended Pacing	
Approximately 30 days	
New Jersey Student Learning Standards for Mathematical Practice	
Standard: Standards for	
CPI #	**Cumulative Progress Indicator (CPI)**
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New Jersey Student Learning Standards for English Language Arts Companion Standards	
Standard: Science Key Ideas and Details	
CPI #	**Cumulative Progress Indicator (CPI)**
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New Jersey Student Learning Standards for Career Readiness, Life Literacies and Key Skills	
CPI #	**Cumulative Progress Indicator (CPI)**
9.4.12.CT.2	Explain the potential benefits of collaborating to enhance critical thinking and problem solving (e.g., 1.3E.12profCR3.a).
New Jersey Student Learning Standards for Computer Science and Design Thinking	
CPI #	**Cumulative Progress Indicator (CPI)**
8.2.12.NT.1	Explain how different groups can contribute to the overall design of a product.
New Jersey Student Learning Standards for Science	
HS-PS2-4	Use mathematical representations of Newton's Law of Gravitation and Coulomb's Law to describe and predict the gravitational and electrostatic forces between objects.

Instructional Focus
Unit Enduring Understandings
● The use of a 2D graph to represent the visual relationship between a function and one independent variable can be extended to the use of 3D graphs to represent the visual relationship between a function and two independent variables. ● It is advantageous to visually represent 3D functions using contour maps and topographical maps that are represented in 2D. ● A derivative of a function is dependent on how one independent variable changes. ● The idea of tangent lines at a specific point of a single-variable function used to approximate values can be extended to using tangent planes at a specific point of a two-variable function to approximate values. ● Multivariable calculus concepts help explain phenomena in the physical world.
Unit Essential Questions
● How can a contour map be used as a tool to describe the rate of change of a function of two variables? ● How do you describe a limit in 3D space? ● Why do partial derivatives hold true for single-variable functions? ● Why are vectors used to describe rates of change of a two-variable function? ● Why is a tangent plane used to approximate a value of a two-variable function?
Content Understandings
● The rate of change of a function of two or more variables is the sum of the partial derivatives of that function with respect to each independent variable. ● The Product, Quotient, and Chain rules from single-variable calculus are extended to multivariable functions. ● A directional derivative is the rate of change of a multivariable function in a specific direction and can be expressed as the dot product of the gradient vector and a unit vector. ● Analyzing the first and second derivatives of two-variable functions can determine critical points and extrema to be used in optimization of two-variable functions. ● The total differential of a function is dependent on the combination of the products of the partial derivatives with respect to each independent variable and the differential of each respective independent variable.
Content Questions
● What are level curves and level surfaces in relation to multivariable functions? ● What does it mean for a function in three-space to be continuous? ● What is a partial derivative, and how is it interpreted? ● How is the chain rule applied when taking derivatives of functions of two or more variables? ● What is the equation of a tangent plane of a function at a point, and how is it found? ● What is a directional derivative? ● What is a gradient vector, and what meaning does it have? ● How does one calculate the minima and maxima values of a function of two variables? ● What are the applications of the maximum or minimum value of a function?
Objectives
Students are learning to/that: ● Analyze with functions of two or three variables. ● Graph functions of two variables using vertical and horizontal traces. ● Graph level curves. ● Describe level surface. ● Define the limit of a function of several variables.

- Find a limit using properties of limits.   - Examine when limits exist.   - Determine where a function is continuous.   - Find the partial derivatives of a function of two variables.   - Interpret partial derivatives as a slope of a tangent line.   - Interpret partial derivatives as a rate of change.   - Find second-order partial derivatives.   - Find the partial derivatives of a function of multiple variables.   - Show that a function of two variables is differentiable.   - Use the differential as an approximating tool.   - Find the differential of a multivariable function.   - Differentiate functions of several variables where each variable is a function of a single variable.   - Differentiate functions of several variables where each variable is a function of two or more variables.   - Differentiate an implicitly defined function of several variables.   - Use chain rule to take derivatives of functions of several variables.   - Find the directional derivative of a multivariable function.   - Find the gradient of a function of two variables.   - Use properties of the gradient vector to analyze a multivariable function.   - Find an equation of a tangent plane to a surface.   - Find an equation of a normal line to a tangent plane.   - Determine the critical points of a function.   - Utilize the Second Partial Derivatives Test to find the absolute extrema of a function of two variables.   - Solve optimization problems.   - Use Lagrange multipliers for an optimization problem with one or more constraints.
Evidence of Learning
Assessment
Assessment plan may include teacher designed formative and summative assessments, a district common assessment, analysis of standardized tests and NJSLA data.
Resources
Core Text: Multivariable Calculus, Early Transcendentals, 2e (2019), Sullivan & Miranda

Unit 4: Multiple Integrals	
Content Area: Mathematics	
Course & Grade Level: Multivariable Calculus Honors, grade 12	
Summary and Rationale	
In single-variable calculus, definite integrals are used to find the area between curves in the 2D plane. Double and Triple integrals extend upon this idea in 3D space. With double integrals, it is still possible to find the area between two curves, but now it is also possible to find volume between surfaces and the area of a bounded surface. Triple integrals allow for the calculation of volume of implicitly defined surfaces as well. This can be achieved in different coordinate systems, including rectangular, cylindrical, spherical, and even general systems. Other applications of multiple integrals can be seen in finding centroids, centers of mass, and moments of inertia.	
Recommended Pacing	
Approximately 25 days	
New Jersey Student Learning Standards for Mathematical Practice	
Standard: Standards for	
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New Jersey Student Learning Standards for English Language Arts Companion Standards	
Standard: Science Key Ideas and Details	
CPI #	**Cumulative Progress Indicator (CPI)**
RST.9-10.3	Follow precisely a complex multistep procedure when carrying out experiments, taking measurements, or performing technical tasks, attending to special cases or exceptions defined in the text. Follow precisely a multistep procedure when carrying out experiments, taking measurements, or performing technical tasks.
New Jersey Student Learning Standards for Career Readiness, Life Literacies and Key Skills	
CPI #	**Cumulative Progress Indicator (CPI)**
9.4.12.CT.2	Explain the potential benefits of collaborating to enhance critical thinking and problem solving (e.g., 1.3E.12profCR3.a).
New Jersey Student Learning Standards for Computer Science and Design Thinking	
CPI #	**Cumulative Progress Indicator (CPI)**
8.2.12.NT.1	Explain how different groups can contribute to the overall design of a product.
Instructional Focus	
Unit Enduring Understandings	
• The idea of using infinitely thin rectangles to find the area between two curves in 2D space can be extended to using infinitely thin rectangular prisms to find the volume between two surfaces. • Utilizing different coordinate systems simplify calculations in multivariable situations.	

- Orientation of curves and surfaces is important for making calculations in multivariable situations.
- Multivariable calculus concepts help explain phenomena in the physical world.

Unit Essential Questions

- How can multiple integrals be used to find area and volume?
- How can multiple integrals be used to find the center of an object?
- Why are different coordinate systems, such as polar (in 2D), cylindrical (in 3D), and spherical (in 3D) important in multivariable calculus?
- How does the orientation of bounded areas and surfaces affect the calculation of area or volume using multiple integrals?
- How do mathematicians benefit from using different coordinate systems?

Content Understandings

- The establishment of double integrals over rectangular regions lays the foundations for calculating double integrals over general regions in 2D space, including rectangular, polar, and general (uv-) coordinate systems.
- Double and triple integrals can be used to explain and fortify concepts learned in single-variable calculus.
- Double and triple integrals can find the center of mass of a lamina or solid.
- The center of mass of a uniformly-dense object is the centroid of that object.
- The differential dA represents an infinitesimally small change in the area, which will be defined differently in each coordinate system.
- The differential dV represents an infinitesimally small change in the volume, which will be defined differently in each coordinate system.
- The Jacobian determinant allows for the transformation between coordinate systems.

Content Questions

- What are the similarities and differences between integration in single-variable calculus and integration in multivariable calculus?
- What are the benefits of changing the coordinate system when computing multiple integrals?
- How are the center of mass and geometric centroids related?
- How are the cylindrical and spherical coordinate systems in 3D established from the polar coordinate system in 2D?
- How are the rectangular, cylindrical, and spherical coordinate systems related to each other in three-dimensional space?
- What is a Jacobian determinant, and how is it established using vectors?

Objectives

Students are learning to/that:

- Determine the area between curves using double integrals.
- Determine the volume between bounded surfaces using double and triple integrals.
- Determine the area of a bounded surface using double integrals.
- Locate the center of mass of an object.
- Find the moment of inertia of a rotating object.
- Establish the cylindrical and spherical coordinate systems in 3D using the polar coordinate system in 2D.
- Convert expressions from one coordinate system to another.
- Set up or rewrite multiple integrals in different coordinate systems.
- Calculate multiple integrals in different coordinate systems.
- Establish the Jacobian determinant.
- Utilize the Jacobian determinant to transform a coordinate system to a simpler, generic coordinate system.

Evidence of Learning

Assessment

Assessment plan may include teacher designed formative and summative assessments, a district common assessment, analysis of standardized tests and NJSLA data.
Resources
Core Text: Multivariable Calculus, Early Transcendentals, 2e (2019), Sullivan & Miranda

Unit 5: Vector Calculus	
Content Area: Mathematics	
Course & Grade Level: Multivariable Calculus, grade 12	
Summary and Rationale	
Vector Calculus ties the concepts of multivariable calculus together. The exploration of movement in vector fields, such as gravitational fields, electric fields, and magnetic fields can be explained through vector calculus. The idea of line and surface integrals and how they relate to multiple integration in scalar fields through Green's Theorem, Stokes' Theorem, and the Divergence Theorem are important for calculating this movement. The distinction between conservative and non-conservative vector fields also contributes to the understanding of physics concepts, such as work and flux.	
Recommended Pacing	
Approximately 25 days	
New Jersey Student Learning Standards for Mathematical Practice	
Standard: Standards for	
CPI #	**Cumulative Progress Indicator (CPI)**
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New Jersey Student Learning Standards for English Language Arts Companion Standards	
Standard: Science Key Ideas and Details	
CPI #	**Cumulative Progress Indicator (CPI)**
RST.9-10.3	Follow precisely a complex multistep procedure when carrying out experiments, taking measurements, or performing technical tasks, attending to special cases or exceptions defined in the text. Follow precisely a multistep procedure when carrying out experiments, taking measurements, or performing technical tasks.
New Jersey Student Learning Standards for Career Readiness, Life Literacies and Key Skills	
CPI #	**Cumulative Progress Indicator (CPI)**
9.4.12.CT.2	Explain the potential benefits of collaborating to enhance critical thinking and problem solving (e.g., 1.3E.12profCR3.a).
New Jersey Student Learning Standards for Computer Science and Design Thinking	
CPI #	**Cumulative Progress Indicator (CPI)**
8.2.12.NT.1	Explain how different groups can contribute to the overall design of a product.
Instructional Focus	
Unit Enduring Understandings	
• The world is made up of vector fields, which can be conservative or non-conservative. The world's gravitational field is conservative, so energy in this field is neither created nor destroyed, justifying the first law of thermodynamics.	

- Line integrals and surface integrals in vector fields are used to describe movement inside the vector field and can be calculated using scalar functions and multiple integrals.
- Orientation of curves and surfaces is important for making calculations in multivariable situations.
- Multivariable calculus concepts help explain phenomena in the physical world.

Unit Essential Questions

- How are vector fields used to describe real world phenomena?
- How can line integrals be used to describe movement of objects in vector fields?
- What are the differences and similarities between scalar fields and vector fields?

Content Understandings

- The line integral relates the concept of Riemann sums for curves in the 2D plane to curves in 3D space.
- The line integral in vector fields describes the work done on an object moving in the vector field.
- A conservative vector field is one that can be defined by a gradient vector function, meaning that there exists a scalar function, called a potential function, that is associated with that specific gradient vector function.
- The fundamental theorem of line integrals relates to the concepts of the fundamental theorem of calculus.
- The fundamental theorem of line integrals establishes independence of path in a conservative vector field.
- Curl and divergence describe the tendencies of an object at a specific point in a vector field.
- Curl and divergence are used as the foundation for Green's Theorem, Stokes' Theorem, and the Divergence Theorem.
- Surfaces can be defined parametrically.
- The relationship between line integrals and surface integrals establish useful theorems for vector fields.

Content Questions

- What are the different ways we can represent vector fields?
- What is a conservative vector field?
- How are line integrals calculated in scalar fields and vector fields?
- What makes a vector field conservative?
- How is the fundamental theorem for line integrals used to calculate line integrals in conservative fields?
- What is curl?
- What is divergence?
- How is Green's Theorem and Stokes' Theorem useful in calculating line integrals?
- How are line integrals and surface integrals related?
- How is the Divergence Theorem useful for calculating flux integrals?

Objectives

Students are learning to/that:

- Draw a vector field and discuss its different representations.
- Calculate line integrals in scalar fields and vector fields.
- Utilize the fundamental theorem for line integrals.
- Determine if a vector field is conservative or non-conservative.
- Find the potential function of a conservative vector field.
- Determine the curl vector of an object at a specific point in a vector field.
- Determine the divergence of an object at a specific point in a vector field.
- Utilize Green's theorem and Stokes' Theorem to calculate line integrals.
- Parameterize surfaces.
- Calculate surface integrals in scalar fields and vector fields.
- Calculate flux integrals.
- Utilize the Divergence Theorem to calculate flux integrals.

Evidence of Learning

Assessment

Assessment plan may include teacher designed formative and summative assessments, a district common assessment, analysis of standardized tests and NJSLA data.
Resources
Core Text: Multivariable Calculus, Early Transcendentals, 2e (2019), Sullivan & Miranda